Recursive Napoleon-Like Constructions Investigated with a Symbolic Geometry System

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Abstract. *We investigate limit behavior for the recursive application of a variety of constructions generalized from that of Napoleon's Theorem. Napoleon's Theorem states that if we draw an isosceles triangle with 120 degree angle at its apex on each of the three sides of a triangle, then the triangle formed by joining those apexes is equilateral. We generalize this construction by allowing any set of similar triangles. In particular we show that if the similar triangles have 120 degree angles at their apex, then recursive application of this pseudo Napoleon's construction will, in the limit, construct a triangle congruent to Napoleon's. We investigate this class of problems with a symbolic geometry system and a Computer Algebra System.*

Introduction

In this paper, we attempt to illustrate the use of a symbolic geometry system in close combination with a computer algebra system. We will illustrate this in an investigation of the limit properties of certain classes of recursive constructions.

These recursive constructions are all in some sense or another generalizations or analogues of Napoleon's Theorem [Coxeter 1967; Hahn 1994]. Napoleon's Theorem has been generalized in a variety of different ways [Gerber 1980; Mauldon 1966; Rigby 1988], and recursive sequences of Napoleon-like constructions studied in [Ziv 2002]. The latter sequences of constructions are similar, but not identical to those of section 4, below.

Geometry Expressions (www.geometryexpressions.com) can take a geometry diagram with symbolic constraints (distances, angles, coordinates, etc.) and generate algebraic expressions for output measurements. This is a functionality which has not been directly available in an interactive sketch based tool, and it is the objective of this paper to illustrate how such a tool may be used.

In this paper, we will accept the formulas output from the symbolic geometry system as correct. The reader with less faith in technology is welcome to prove these for himself.

1. A Penequilateral Triangle

Given a triangle ABC (not necessarily right), we draw a Pythagoras-like diagram by subtending squares on each side. We construct a second triangle DEF by joining the centers of the segments between neighboring corners of the squares (fig. 1).

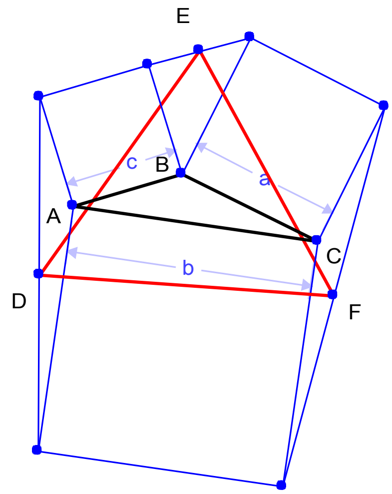

Figure 1: Construction for a penequilateral triangle

Although DEF can look to be equilateral at first glance, it is not. Its side lengths are shown in figure 2.

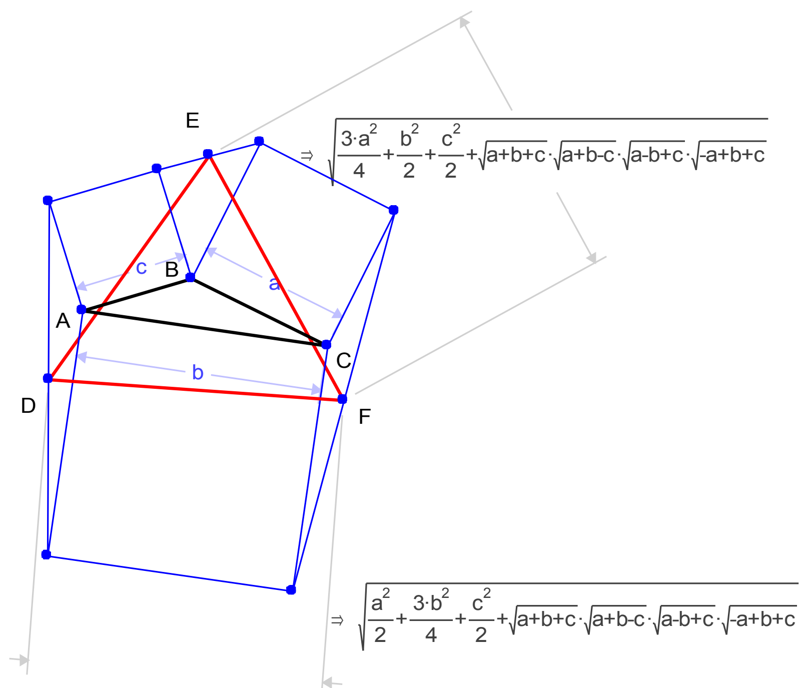

Figure 2: side lengths of a penequilateral triangle

Clearly the difference in squares of the side lengths of the new triangle is:

$$\frac{a^2 - b^2}{4}$$

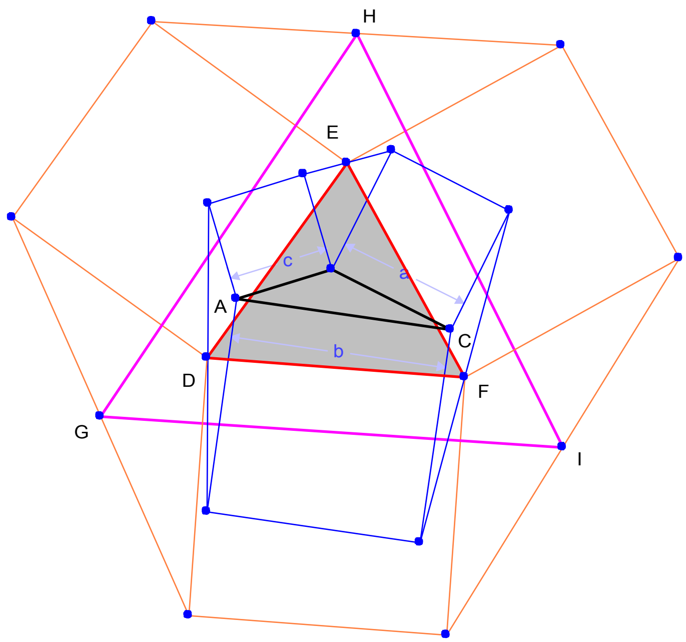

Figure 3: recursive application of the construction

Applying the construction to DEF will result in a third triangle GHI, whose corresponding sides will have a difference of squares of:

$$\frac{a^2 - b^2}{16}$$

If we keep recursing this construction, therefore the difference in squares of side length tends to zero.

We denote S(i) to be the sum of squares of the side lengths of the triangle at step i in our recursion, and A(i) to be its area. Let ABC be triangle 0, and DEF triangle 1. Then:

$$S(0) = a^2 + b^2 + c^2$$

$$A(0) = \frac{\sqrt{a+b+c}\sqrt{a+b-c}\sqrt{a-b+c}\sqrt{-a+b+c}}{4} \quad \text{(Heron's formula)}$$

And from figure 2

$$S(1) = 12A(0) + \frac{7}{4}S(0)$$

Examining the expression for the area of DEF (figure 4), we see that A(1) can also be written as a linear combination of A(0) and S(0):

$$A(1) = \frac{7}{4}A(0) + \frac{1}{4}S(0)$$

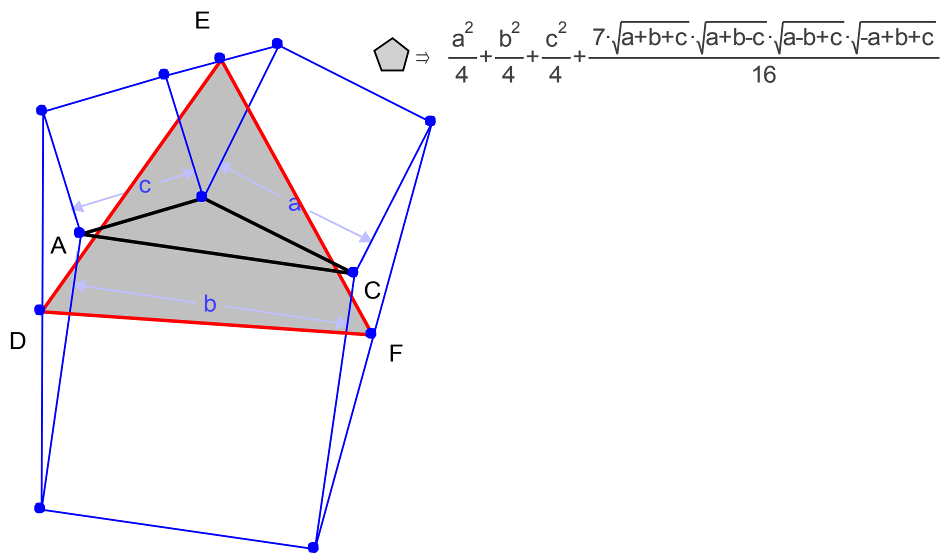

Figure 4: area of the constructed triangle

Hence if

$$M = \begin{bmatrix} \frac{7}{4} & \frac{1}{4} \\ 12 & \frac{7}{4} \end{bmatrix}$$

Then

$$\begin{bmatrix} A(n) & S(n) \end{bmatrix} = M \begin{bmatrix} A(n-1) \\ S(n-1) \end{bmatrix}$$

and

$$\begin{bmatrix} A(n) & S(n) \end{bmatrix} = M^n \begin{bmatrix} A(0) \\ S(0) \end{bmatrix}$$

The eigenvalues of M are:

$$\frac{7}{4} + \sqrt{3}, \frac{7}{4} - \sqrt{3}$$

As the largest eigenvalue is greater than 1, M^n tends to infinity, and our triangle grows bigger and bigger. However, the difference in squares of the sides tends to zero, hence the triangle itself tends towards equilateral.

2. An Asymptotically Equilateral Triangle

We next consider a second, similar recursive construction. Given a triangle ABC, we draw equilateral triangles on each side (Napoleon's diagram). We create a new triangle DEF by joining the midpoints of the line segments between the apexes of these triangles (figure 5)

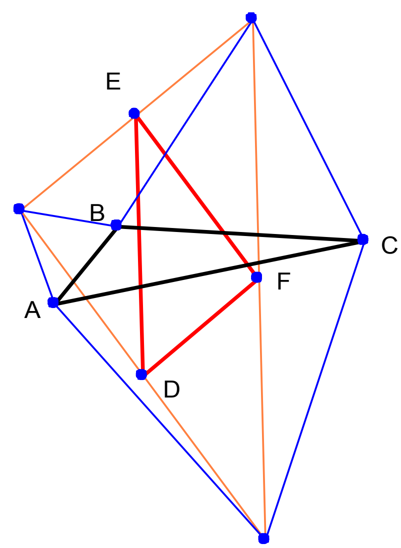

Figure 5: Triangle DEF has vertices the midpoints of the segments joining the apexes of the equilateral triangles drawn on the sides of the generating triangle ABC

Again we consider the effect of recursive application of this construction.

Examining expressions for the length of a side and the area of the triangle, we see that this construction again operates linearly on the sum of squares of the side lengths and the area of the triangle.

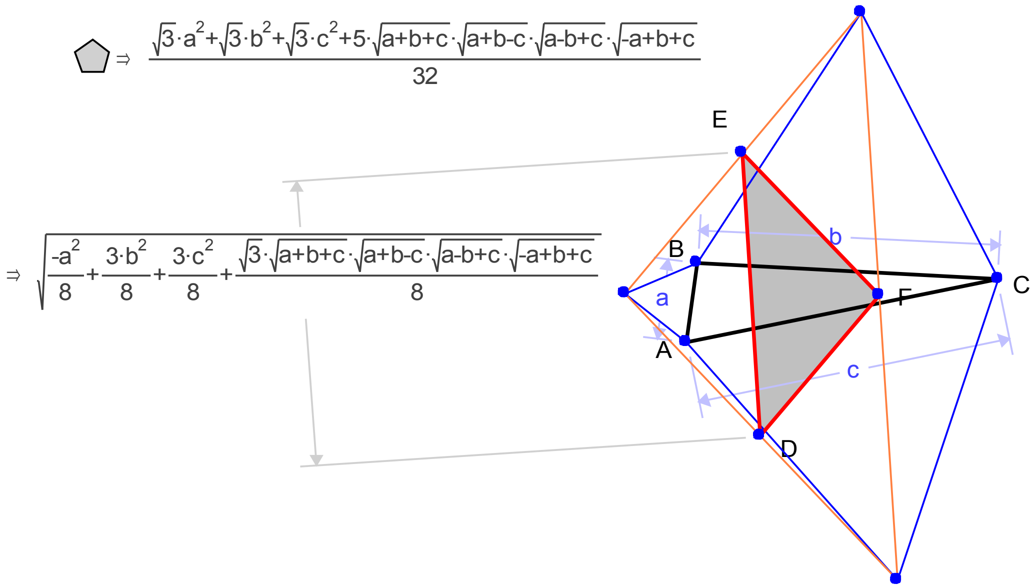

Figure 6: Side length and area of DEF

In fact we have the following:

$$A(1) = \frac{5}{8}A(0) + \frac{\sqrt{3}}{32}S(0)$$

$$S(1) = \frac{3\sqrt{3}}{2}A(0) + \frac{5}{8}S(0)$$

Hence if

$$M = \begin{bmatrix} \frac{5}{8} & \frac{\sqrt{3}}{32} \\ \frac{3\sqrt{3}}{2} & \frac{5}{8} \end{bmatrix}$$

$$\begin{bmatrix} A(n) & S(n) \end{bmatrix} = M^n \begin{bmatrix} A(0) \\ S(0) \end{bmatrix}$$

The eigenvalues of M are:

$$1, \frac{1}{4}$$

The eigenvectors corresponding to these eigenvalues are:

$$\begin{bmatrix} \frac{1}{7} & \frac{4\sqrt{3}}{7} \end{bmatrix}, \begin{bmatrix} -\frac{1}{7} & \frac{4\sqrt{3}}{7} \end{bmatrix}$$

Expressed in terms of the eigenvectors, the matrix M is :

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$M^n = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4^n} \end{bmatrix}$$

$$\lim_{n \to \infty} M^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence

$$\lim_{n \to \infty}([A(n), Sn]) \propto \begin{bmatrix} \frac{1}{7} & \frac{4\sqrt{3}}{7} \end{bmatrix}$$

and

$$\lim_{n \to \infty} \frac{A(n)}{S(n)} = \frac{\sqrt{3}}{12}$$

We show that for a triangle with side lengths a,b,c,

$$\frac{A}{S} \leq \frac{\sqrt{3}}{12}$$

With equality only when a=b=c.

Proof: for an equilateral triangle of side length a

$$A = \frac{\sqrt{3}a^2}{4}$$

$$S = 3a^2$$

Hence

$$\frac{A}{S} = \frac{\sqrt{3}}{12}$$

Now in the general case we have:

$$\frac{A^2}{S^2} = \frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{16(a^2+b^2+c^2)^2}$$

We examine

$$3S^2 - 144A^2 = 192(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)$$

And show this is bigger than or equal to zero with equality only when a=b=c:

Hence:

$$P = \frac{3S^2 - 144A^2}{192} = a^2(a^2-b^2) + b^2(b^2-c^2) - c^2(a^2-c^2)$$

Assume:

$$a \geq b \geq c$$

Then:

$$P \geq b^2(a^2-b^2) + b^2(b^2-c^2) - c^2(a^2-c^2) = b^2(a^2-c^2) - c^2(a^2-c^2) \geq 0$$

with equality only when a=b=c.

With this result, we have shown that the limit triangle of the recursive construction is equilateral.

To derive its side length, we express [A,S] in terms of the eigenvectors:

$$\begin{bmatrix} A & S \end{bmatrix} = \left(\frac{S}{2} + 2\sqrt{3}A\right)\begin{bmatrix} \frac{1}{7} & \frac{4\sqrt{3}}{7} \end{bmatrix} + \left(\frac{S}{2} - 2\sqrt{3}A\right)\begin{bmatrix} -\frac{1}{7} & \frac{4\sqrt{3}}{7} \end{bmatrix}$$

Hence, the quantity $\frac{S}{2} + 2\sqrt{3}A$ is conserved, and the side length **d** of the limit equilateral triangle can be derived from:

$$3d^2 = \frac{S}{2} + 2\sqrt{3}A$$

$$d = \sqrt{\frac{S}{6} + \frac{2}{\sqrt{3}}A},$$

which is the length of the side of the Napoleon’s triangle.

We have proved the following:

Theorem: Define a series of triangles $T_0, T_1, T_2, ..$ such that T_n is derived from T_{n-1} by constructing equilateral triangles on the sides of T_{n-1}, and joining the bisectors of the segments joining the apexes of these equilateral triangles. Then the limit of T_n as n tends to infinity is congruent to the Napoleon triangle of the original triangle.

Proof: follows directly from the above.

Figure 7 is a picture of the first 3 iterations of the construction

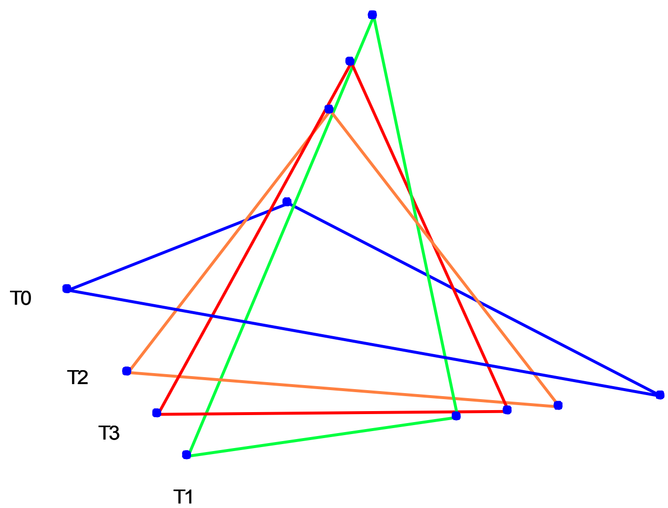

Figure 7: The first 3 iterations of the construction

Denote N(T) the napoleon triangle of triangle T. We have shown that each stage of the recursion creates a triangle whose Napoleon Triangle is congruent to the Napoleon triangle of the original triangle.

$$N(T_0) \cong N(T_1) \cong N(T_2) \cong ..$$

In fact we can show that the construction preserves not just the size of the Napoleon triangle, but also its location.

In figure 8, GH are the constructed points of the new triangle. K is the center of the equilateral triangle on GH. We see from the expression for the coordinates of K that it is also the center of the equilateral triangle BCD.

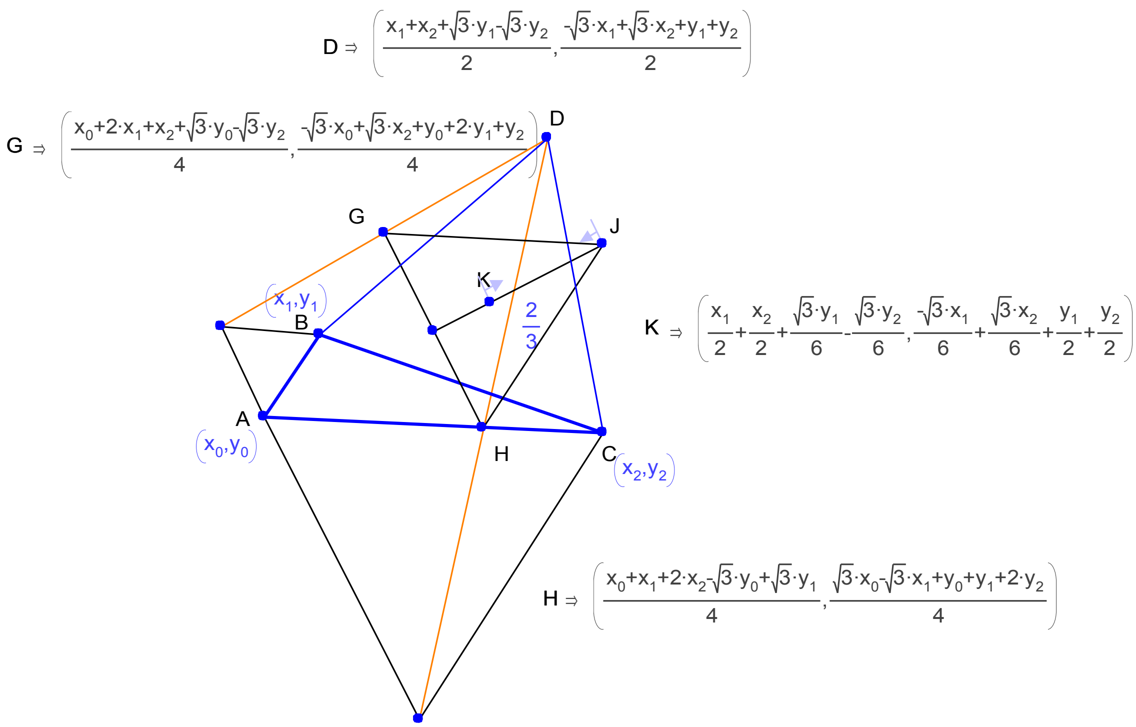

Figure 8: the center of the equilateral triangle on GH is identical to the center of the triangle on BC

Hence we have:

$$N(T_0) = N(T_1) = N(T_2) = ..$$

Hence this construction preserves not only the size, but also the location of the Napoleon triangle.

Hence T_∞ is equilateral with

$$N(T_\infty) = N(T_0)$$

But for equilateral T,

$$N(N(T))=T$$

Hence :

$$T_\infty = N(N(T_0))$$

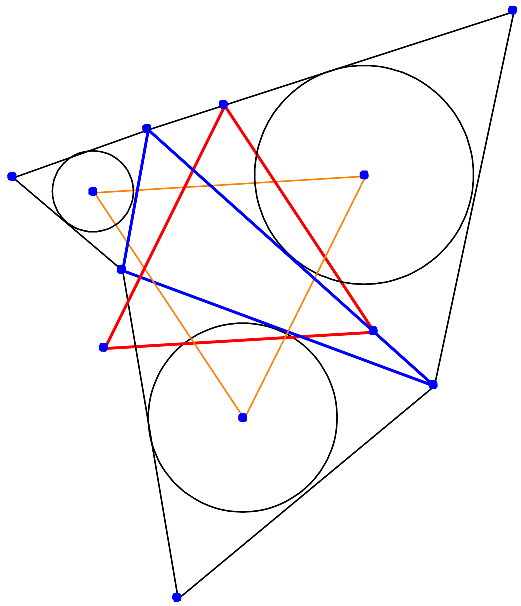

Figure 9: The limit of the recursive construction is the Napoleon's triangle of the Napoleon's triangle of the generator.

3. A Recursive Pseudo Napoleon Construction

A generalization of Napoleon's Theorem states that drawing any similar triangles on the edges of a generating triangle, in such a way that their orientation permutes, then joining any equivalent point on those triangles gives a similar triangle. We investigate the construction where similar triangles are drawn in the same orientation, and where we join the apexes of those triangles. We examine the recursive behavior of this construction. Napoleon's theorem is the special case where the similar triangles have angles 30, 120, 30.

First we prove the following:

Lemma: The centroid of the constructed triangle is the centroid of the original triangle

Proof: If we parametrize the triangle such that the altitude is **t** times the base length, and the foot of the altitude is proportion **s** along the base. Then if a triangle has points (x_0, y_0), (x_1, y_1), (x_2, y_2) The apex has coordinates:

$$\left((1-s)x_0 + sx_1 + t(y_0 - y_1), (1-s)y_0 + sy_1 + t(x_0 - x_1)\right)$$

Clearly, averaging the three symmetric points gives a centroid $\left(\frac{x_0}{3}+\frac{x_1}{3}+\frac{x_2}{3}, \frac{y_0}{3}+\frac{y_1}{3}+\frac{y_2}{3}\right)$, which is identical to that of the original triangle.

We again investigate the recursive behavior of this construction by examining side lengths and areas:

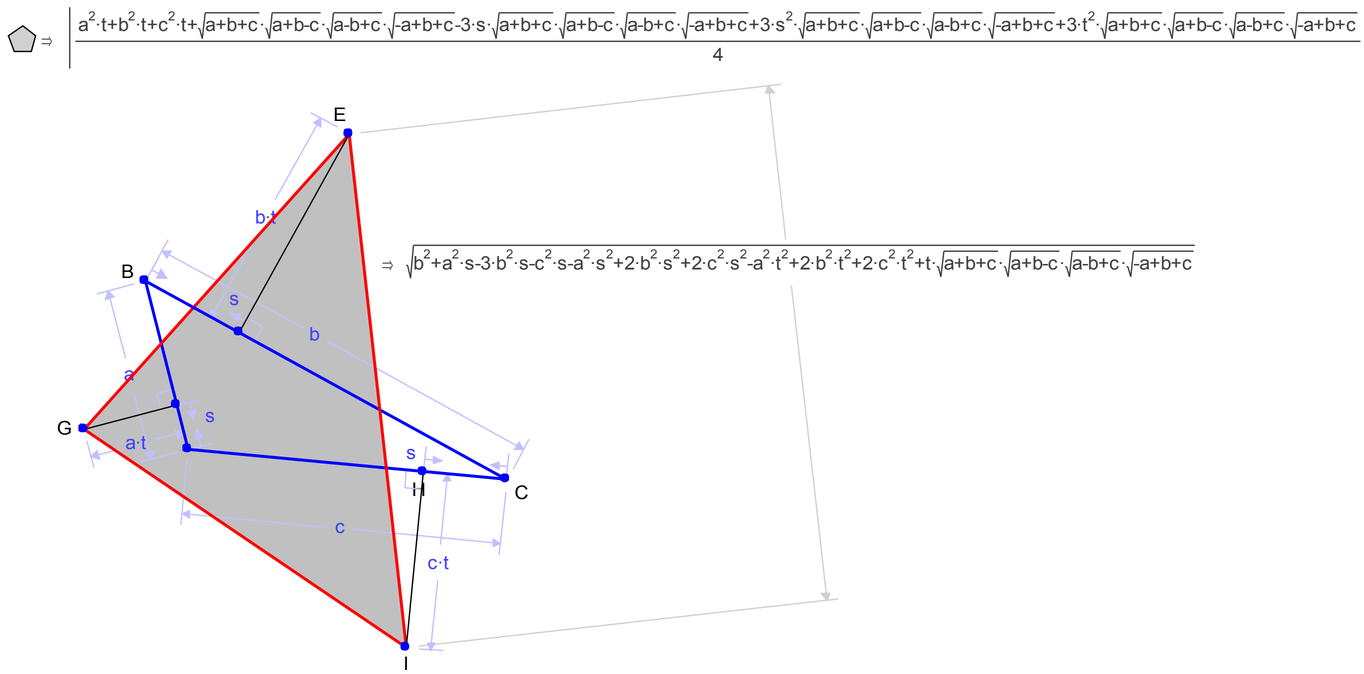

Figure 10: Area and side length for the general triangle apex construction

In the notation of the previous sections, we have:

$$A(1) = \left(1 - 3s + 3s^2 + 3t^2\right)A(0) + \frac{t}{4}S(0)$$

$$S(1) = 12tA(0) + \left(1 - 3s + 3s^2 + 3t^2\right)S(0)$$

Hence if

$$M = \begin{bmatrix} \left(1 - 3s + 3s^2 + 3t^2\right) & \frac{t}{4} \\ 12t & \left(1 - 3s + 3s^2 + 3t^2\right) \end{bmatrix}$$

$$\begin{bmatrix} A(n) & S(n) \end{bmatrix} = M^n \begin{bmatrix} A(0) \\ S(0) \end{bmatrix}$$

The eigenvalues of M are:

$$1 - 3s + 3s^2 + 3t^2 + \sqrt{3}t, 1 - 3s + 3s^2 + 3t^2 - \sqrt{3}t$$

The limit behavior of the construction can be described as follows:

$1-3s+3s^2+3t^2+\sqrt{3}t>1$ constructed triangle grows to infinite size

$1-3s+3s^2+3t^2+\sqrt{3}t=1$ constructed triangle stays finite

$1-3s+3s^2+3t^2+\sqrt{3}t<1$ constructed triangle shrinks to zero

In any case, the eigenvectors are again:

$$\begin{bmatrix}\frac{1}{7} & \frac{4\sqrt{3}}{7}\end{bmatrix}, \begin{bmatrix}-\frac{1}{7} & \frac{4\sqrt{3}}{7}\end{bmatrix}$$

And hence the limiting triangle is equilateral.

To characterize the constructions from this family which stay finite, we solve

$$1-3s+3s^2+3t^2+\sqrt{3}t=1$$

For t:

$$t=\frac{1}{6}\sqrt{3+36s-36s^2}-\frac{\sqrt{3}}{6}$$

We can verify from this expression that s and t lie on the circle:

$$\left(s-\frac{1}{2}\right)^2+\left(t+\frac{\sqrt{3}}{6}\right)^2=\frac{1}{3}$$

This is the circle through the points (0,0), (0,1), $\left(\frac{1}{2},\frac{\sqrt{3}}{6}\right)$.

The latter is the center of an equilateral triangle drawn on the segment (0,0), (1,0). Hence the chord (0,0), (1,0) subtends an angle of 120 degrees with any point (with positive t) on this arc.

Hence the limit behavior of the construction depends on the angle at the apex of the similar triangles. If this angle is less than 120 degrees the constructed triangle grows infinitely. If this is more than 120 degrees, the constructed triangle shrinks. If this equals 120 degrees the constructed triangle stays finite, and using the argument of section 3 tends to a triangle congruent to the Napoleon's triangle.

Looking at the smaller eigenvalue above, we find a value of t which makes this eigenvalue 0:

$$t=\frac{\sqrt{3}}{6}+i\left(\frac{1}{2}-s\right)$$

t is real only when $s=\frac{1}{2}$, in which case this is Napoleon's construction.

Note that referred to an axis aligned with the eigenvectors, in the case of Napoleon's construction:

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

As we know, Napoleon's construction takes us directly to the limiting configuration.

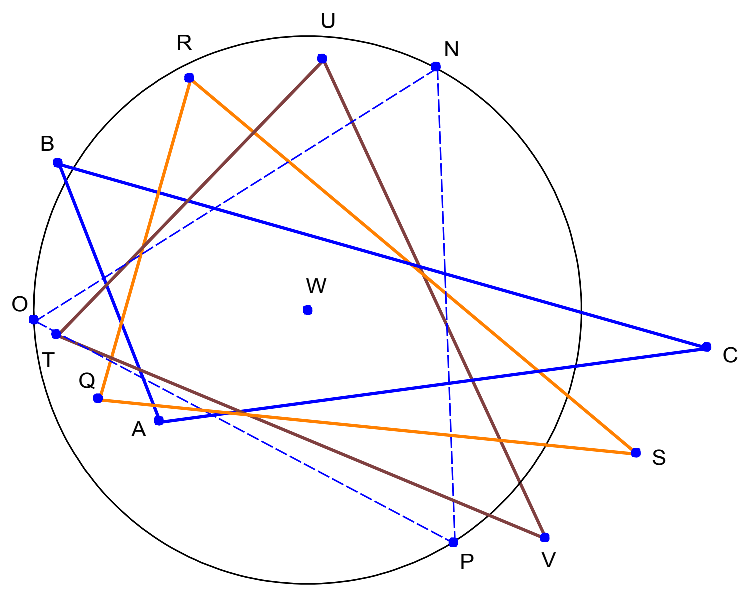

Fig 11 ABC is the initial triangle. NOP is its Napoleon's Triangle. QRS, TUV are the first two iterations of the construction

Although the size and shape of the triangle tends to a limit, its location does not in this case. However, we can say the following:

Theorem: Define a series of triangles $T_0, T_1, T_2, ..$ such that T_n is derived from $T_{n\text{-}1}$ by constructing similar triangles on the sides of $T_{n\text{-}1}$, whose opposite angle is 120 degrees and joining the apexes of these triangles. Then as n tends to infinity T_n tends to be an equilateral triangle with the same circumcircle as the Napoleon's Triangle of the original triangle..

Proof: Referred to axes aligned with the eigenvectors, we see that applying Napoleon's construction to T_n has a matrix of

$$N \cdot M^n = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \lambda_2{}^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is the same matrix as applying Napoleon's construction to T_0.

Hence $N(T_n)$ is congruent to that of $N(T_0)$. (By the lemma, $N(T_n)$ has the same circumcircle as $N(T_0)$)

As T_n tends to equilateral, the limiting form is an equilateral triangle whose centroid is the same as T_0 and whose Napoleon's triangle is congruent to that of T_0. But an equilateral triangle is congruent to its Napoleon's triangle, hence the result.

5. Conclusions

Numeric Dynamic Geometry Software can be used in the discovery of theorems where collinearity or congruence, or extremely simple algebraic relations hold between measured quantities, as, for example, in [Grunbaum 2001]. A symbolic geometry package, such as Geometry Expressions allows a much richer discovery process where the form of algebraic expressions derived automatically from the geometry are the catalysts for mathematical development. Specifically, the sum of squares of the lengths of the sides of the triangles and their areas at each stage in the iteration were shown to be linearly related with those of the triangle at the previous stage in the iteration. This let the mechanics of linear algebra be applied to the problem of acquiring a limiting form.

In this paper, we have taken the more radical approach, not only of using the symbolic geometry system as a means of discovering mathematics, but also of accepting its results without proof. We have no excuse for this other than to maintain that these proofs while tedious are not difficult to do by hand.

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